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HEAT CONDUCTION EQUATION FOR SYSTEMS WITH AN INHOMOGENEOUS INTERNAL STRUCTURE

S. L. Sobolev

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Based on the two-temperature model, a heat conduction equation for inhomogeneous systems is suggested. The conditions under which this equation is reduced to a classical transfer equation of the parabolic type or a local-nonequilibrium transfer equation of the hyperbolic type are discussed. The parameters characterizing heat transfer in an inhomogeneous medium are discussed from a physical viewpoint.

The classical theory of heat transfer processes gives a heat conduction equation of the parabolic type, which is a consequence of the energy conservation law and the Fourier law $\mathbf{q} = -\lambda \nabla T$. It is known that a parabolic type heat conduction equation leads to the physically incorrect conclusion of an infinite propagation velocity of thermal disturbances – a change in the temperature at a given point of space is instantaneously manifested at an infinitely remote point. This indicates that a parabolic type heat conduction equation cannot be used to describe high-speed processes. For this, account should be taken of the local nonequilibrium of the transfer process, which in a simple case leads to the modified Fourier law $\mathbf{q} + \tau \partial \mathbf{q}/\partial t = -\lambda \nabla T$, where τ is the time of system relaxation to the local thermodynamic equilibrium. The modified Fourier law and the law of energy conservation yield, in turn, a heat conduction equation of the hyperbolic type with a finite velocity of propagation of disturbances. The local-nonequilibrium theory of transfer processes is a matter of of interest in numerous works (see the cited works in [1-5]). Note that analogous problems arise in relaxation filtration theory [6-8].

On the other hand, a classical heat conduction equation of the parabolic type is obtained in the continuum approximation, implying that the given system possesses no internal structure. The continuum approximation is violated if the characteristic scale of the transfer process becomes comparable to the scale of an internal structure of the system. This is the case, e.g., in highly rarefied gases, capillary-porous bodies, polymers, pastes, suspensions, cellular systems, liquid crystals, etc. (see the citations in [1-6, 9, 10]).

Transfer processes in systems with an inhomogeneous internal structure may be described using spatially nonlocal models such as models of systems with a space "memory" [3, 4, 11, 12] or models of systems with a discrete structure [3, 4, 12, 13].

In a number of works [5, 9, 10], for investigation of heat conduction processes in inhomogeneous media the use of a hyperbolic-type equation that accounts for system relaxation to a local equilibrium state is suggested. Based on such an approach, estimates of the relaxation time for inhomogeneous systems are obtained and their values are shown to exceed by many orders of magnitude the times of relaxation of gases, liquids, and solids to local equilibrium [5, 9, 10]. Consequently, the relaxation time of a heat transfer process in inhomogeneous media has a different physical meaning. This is also valid for other thermophysical characteristics of an inhomogeneous system. But a question arises: how are the averaged values of thermophysical parameters of an inhomogeneous system related to the thermophysical parameters of its constituent subsystems? Furthermore, till now the domains of applicability of parabolic, hyperbolic, or other types of equations for the description of transfer processes in inhomogeneous media are not specified clearly. In the present work we suggest a heat transfer equation for inhomogeneous media based on the two-temperature model [4, 6, 12, 14-17] that permits one to answer the questions formulated .

We shall consider an inhomogeneous system consisting of two interpermeable continua (homogeneous subsystems). If the time for establishing equilibrium in the subsystems (or in one of them) is much shorter than

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the time for reaching equilibrium between them, then the proper temperatures T_1 and T_2 may be assigned to each subsystem. Such a situation may occur in heterogeneous media (e.g., rocks [14]), in plasma, where electrons and ions play the role of subsystems with different temperatures, in metals (electrons–lattice) [15], in turbulent flows (turbulent spots–laminar interlayers) [6], in filtration combustion [16], and in heterogeneous catalysis [17] (gas–solid). If the time of relaxation of the individual subsystems to the local thermodynamic equilibrium is comparable, in order of magnitude, to the characteristic time of the transfer process, then such a system may be described by a pair of coupled equations of the hyperbolic type [4]. We shall consider sufficiently slow processes, whose characteristic times are much larger than the relaxation times inside the subsystems. In this case, the space-time distributions of temperatures in the subsystems obey a pair of coupled equations of the parabolic type:

$$c_1 \frac{\partial T_1}{\partial t} = \lambda_1 \Delta T_1 + W_1 + g(T_2 - T_1), \tag{1}$$

$$c_2 \frac{\partial T_2}{\partial t} = \lambda_2 \Delta T_2 + W_2 + g(T_1 - T_2), \tag{2}$$

where W_i is the intensity of heat release sources. Note that all the coefficients in (1), (2) are based on a unit volume of the inhomogeneous medium.

In the majority of cases it is more convenient to use an average temperature, which is a measure of the thermal energy per unit volume of the inhomogeneous medium. For two-component system (1), (2) the average temperature has the following form [14]: $\overline{T} = (c_1T_1 + c_2T_2)(c_1 + c_2)^{-1}$. From (1), (2) we may derive equations describing space-time evolution of both the subsystems temperatures T_i ($i = 1, 2$) and the average temperature T:

$$\frac{\partial T_i}{\partial t} + \tau^* \frac{\partial^2 T_i}{\partial t^2} - l_a^2 \frac{\partial}{\partial t} \Delta T_i = a^* \Delta T_i - a_g l_g^2 \Delta^2 T_i +$$

$$+ \frac{W_1 + W_2}{c_1 + c_2} + \tau^* \frac{\partial}{\partial t} \frac{W_i}{c_i} - \frac{l_g^4}{c_i} \Delta \frac{W_i}{h_i^2}, \tag{3}$$

$$\frac{\partial \overline{T}}{\partial t} + \tau^* \frac{\partial^2 \overline{T}}{\partial t^2} - l_a^2 \frac{\partial}{\partial t} \Delta \overline{T} = a^* \Delta \overline{T} - a_g l_g^2 \Delta^2 \overline{T} +$$

$$+ \frac{W_1 + W_2}{c_1 + c_2} + \tau^* \frac{\partial}{\partial t} \frac{W_1 + W_2}{c_1 + c_2} - \frac{l_g^4}{c_1 + c_2} \Delta \left(\frac{W_1}{h_1^2} + \frac{W_2}{h_2^2} \right), \tag{4}$$

where $\tau^* = \tau_1\tau_2/(\tau_1 + \tau_2) = c_1c_2/g(c_1 + c_2)$; $\tau_i = c_i/g$; $h_i^2 = \tau^* a_i$; $a_i = \lambda_{ii}/c_i$; $l_a^2 = h_1^2 + h_2^2$; $l_g^2 = h_1h_2$; $a^* = (\lambda_1 + \lambda_2)/(c_1 + c_2)$; $a_g = l_g^2/\tau^*$.

Heat conduction equations for an inhomogeneous medium (3), (4) differ substantially both from a heat conduction equation of the parabolic type and from a heat conduction equation of the hyperbolic type. They contain parameters of temporal τ^* and spatial l nonlocality. This indicates that if the characteristic space-time scales $\overset{*}{\tau}$ and L of the transfer process in an inhomogeneous system are such that $\overset{*}{\tau} \sim \tau^*$ and $L \sim l$, then a local equation of transfer of the parabolic type no longer holds and instead of it nonlocal equations (3), (4) should be used.

We now consider some particular cases of Eqs. (3), (4) and the physical meaning of the parameters entering them. In dissipation processes, when heat is transferred only by diffusion, the macroscales L and $\overset{*}{t}$ of such a process are interrelated by $L^2 = \overset{*}{a}\overset{*}{t}$. In this case, Eqs. (3), (4) in a zero approximation are reduced to classical local equations of the parabolic type, i.e., diffusion equations. In the next approximation, all terms in Eqs. (3), (4) must be taken into consideration, with the heat transfer process in this case being nonlocal. The space nonlocality is characterized by the constants l_a and l_g, which are the arithmetic and geometric mean, respectively, between the characteristic depths of heating of the subsystems h_1 and h_2 for the time τ^*. The time nonlocality is characterized by the parameter τ^*, which is the relaxation time of the temperature difference between the subsystems, i.e., the time of equalization of temperatures or the characteristic time of heat transfer between the subsystems. The parameter τ^* (see above) depends on the characteristic times of the subsystems $\tau_1 = c_1/g$ and $\tau_2 = c_2/g$. Both approximations produce equations with an infinitely high propagation velocity for disturbances, i.e., from the solution of these equations it follows that a change in the temperature at some point of space instantaneously entails a change in the temperature even in an infinitely remote region. From physical considerations it is clear that the propagation velocity of thermal disturbances has a finite, even though very high, value [1-5]. However, in the case of heat transfer by diffusion, where the characteristic velocity of the transfer process is much lower than the propagation velocity of thermal disturbances, parabolic-type equations (3), (4) and their variants for particular cases describe transfer processes with sufficient accuracy.

If the characteristic velocity $V \sim L/t^*$ of the process under consideration, e.g., the velocity of motion of a heat release source or the propagation velocity of a heated zone, is comparable to the characteristic velocity $v_T \sim l/\tau^*$ determined by the ratio of the scales of space and time nonlocality, then from (3), (4) a hyperbolic-type equation [12] follows (we consider the one-dimensional variant):

$$\frac{\partial \overline{T}}{\partial t} + \tau^* \frac{\partial^2 \overline{T}}{\partial t^2} = a^* \frac{\partial^2 \overline{T}}{\partial x^2} + \frac{W_1 + W_2}{c_1 + c_2} + \tau^* \frac{\partial}{\partial t} \frac{W_1 + W_2}{c_1 + c_2} . \tag{5}$$

In this case, there is no space nonlocality for the heat transfer process, and the parameter τ^* characterizes the inertial properties of heat transfer and determines the propagation velocity of thermal disturbances $v_T = (a^*/\tau^*)^{1/2} = ((\lambda_1 + \lambda_2)g/c_1 \cdot c_2)^{1/2}$ (see [1-5, 18]). The presence of the derivative of the source function in (3)-(5) is explained by the inertia of heat transfer in inhomogeneous media, but superficially the thermal impact is perceived by the system with a delay by the characteristic time τ^*. It is noteworthy that the analogy between the relaxation time τ^* and the inertial mass is also confirmed by the relation of τ^* to τ_1 and τ_2. Indeed, the relaxation time τ^* for an inhomogeneous medium described by two-temperature model (1), (2) coincides with the determination of the reduced mass in a system of two bodies. The use of heat conduction equation of the hyperbolic type (5) for the analysis of transfer processes in a medium with an inhomogeneous internal structure gives the best fit of theory and experiment in some cases [5]. Moreover, Eq. (5) may be used in investigations of various homogeneous systems under extreme conditions, i.e., for high gradients and flows, low temperatures, supershort energy pulses, high velocities of running waves, etc. [1-4, 18].

Hyperbolic-type heat conduction equation (5) indicates that a surface of strong discontinuity produced by a temperature jump propagates with a constant velocity as a zero-thickness surface. Furthermore, motion of a heat source with the velocity $V \geq v_t$ induces a temperature jump in the heat release zone, i.e., thermal shock waves [3, 4, 18].

In the next approximation, for regimes of heat transfer with the characteristic velocity $V \sim v_t$, Eqs. (3), (4) acquire the form

$$\frac{\partial \overline{T}}{\partial t} + \tau^* \frac{\partial^2 \overline{T}}{\partial t^2} - l_a \frac{\partial^3 \overline{T}}{\partial x^2 \partial t} = a^* \frac{\partial^2 \overline{T}}{\partial x^2} + \frac{W_1 + W_2}{c_1 + c_2} +$$

$$+ \tau^* \frac{\partial}{\partial t} \frac{W_1 + W_2}{c_1 + c_2} + \frac{l_g^4}{c_1 + c_2} \frac{\partial^2}{\partial x^2} \left(\frac{W_1}{h_1^2} + \frac{W_2}{h_2^2} \right) . \tag{6}$$

In this case the existence of space nonlocality, i.e., an additional, compared to Eq. (5), derivative $\partial^3 T/\partial x^2 \partial t$, results in smoothing of the temperature jump and emergence of a transient layer instead of the discontinuity, whose thickness is proportional to $\sqrt{x}$, where x is the distance from the place where the discontinuity originates [2, 19]. This smoothing of temperature jumps is analogous to the effect of viscosity in gas dynamics [19].

Temperature shock waves, i.e., temperature jumps occurring when heat sources move with the velocity $V \sim v_{tT}$ in a relaxing medium with $l_a = 0$ [3, 4, 18], will also undergo a smoothing action of the spatial nonlocality – "thermal viscosity." In this case, the thermal shock front (temperature discontinuity) will be transformed into a transient structure, i.e., a region of rapid, but continuous change in temperature, concentrated in the vicinity of the former discontinuity. The characteristic dimension of this region for $V = v_t$ is l_a. It is pertinent to note that the fictitious viscosity method, i.e., introduction of a fictitious viscosity term into the equations of gas dynamics, is used to overcome difficulties encountered in the numerical analysis of discontinuous solutions. Moreover, owing to it there is no need to prescribe boundary conditions at the discontinuity in the course of calculation. When the calculation is completed, the fictitious viscosity tends to zero and the gas becomes nonviscous again. Introduction of a fictious "temperature" viscosity may be used for numerical modeling of thermal shock waves [3, 4, 18].

Thus, an analysis of a two-temperature model consisting of coupled local heat conduction equations (1) and (2) shows that the process of heat transfer in a heterogeneous medium is nonlocal. In this sense, the two-temperature model is a link between the classical local theory and various nonlocal models. On the other hand, it is a microscopic model since Eqs. (1) and (2) describe a change in the temperature not of the system as a whole but of its individual subsystems. At the same time this model gives Eqs. (4)-(6) for the average macroscopic temperature, and therefore it allows the microscopic parameters of the system to be related to its average thermodynamic constants. The difference in the types of the microscopic equations entering the two-temperature model confirms once more that the hierarchy of space-time scales in dynamic systems is a problem of paramount importance. Equations (3)-(6) may be employed to study transfer processes in various substances with an inhomogeneous structure both of natural origin, e.g., in limestones and sandstones, and of artificial origin, e.g., in polymers, foam plastics, heterogeneous catalysts, composite materials, liquid crystals, suspensions, pastes, and so on. Moreover, these equations describe transfer processes in systems consisting of two interacting subsystems, heat transfer between which plays an important role, e.g., in plasma (electrons–ions), in metals (electrons–lattice), in highly excited gases (different degrees of freedom of the molecules), in gases with chemical reactions, etc.

NOTATION

q, heat flux; T, temperature; T_i $(i = 1, 2)$, temperature of the subsystems; $\overline{T}$, average temperature; λ, thermal conductivity; τ, relaxation time; c_i, heat capacity of the i-th subsystem per unit volume of the inhomogeneous medium; g, heat transfer coefficient between the subsystems; $\tau^* = c_1 c_2 / g(c_1 + c_2) = \tau_1\tau_2/(\tau_1 + \tau_2)$, relaxation time for the inhomogeneous medium; λ_i, thermal conductivity of the i-th subsystem per unit volume of the inhomogeneous medium; $\tau_i = c_0/g$, characteristic time of heating of the i-th subsystem; $l_a^2 = h_1^2 + h_2^2$, $l_g^2 = h_1 h_2$, nonlocality scales of the inhomogeneous medium; $h_i = \sqrt{\tau^* a_i}$, heating depth of the i-th subsystem for the time τ^*; $a_i = \lambda_i / c_i$, thermal diffusivity of the i-th subsystem; a^*, thermal diffusivity of the inhomogeneous medium; t^* and L, characteristic time and space scales of the transfer process; V, characteristic velocity of the transfer process; v_t, propagation velocity of the heat wave.

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